9-2 Logarithms and Logarithmic Functions

Main Ideas

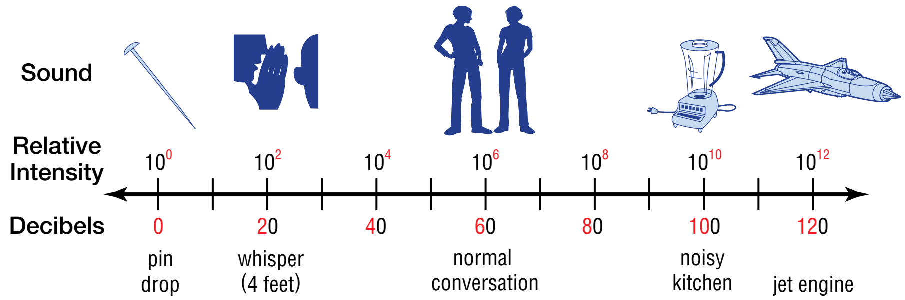
- Evaluate logarithmic expressions.
- Solve logarithmic equations and inequalities.

New Vocabulary

logarithm
logarithmic function
logarithmic equation
logarithmic inequality

GET READY for the Lesson

Many scientific measurements have such an enormous range of possible values that it makes sense to write them as powers of 10 and simply keep track of their exponents. For example, the loudness of sound is measured in units called decibels. The graph shows the relative intensities and decibel measures of common sounds.



The decibel measure of the loudness of a sound is the exponent or logarithm of its relative intensity multiplied by 10.

Review Vocabulary

Inverse Relation
when one relation contains the element (*a*, *b*), the other relation contains the element (*b*, *a*) (Lesson 7-6)

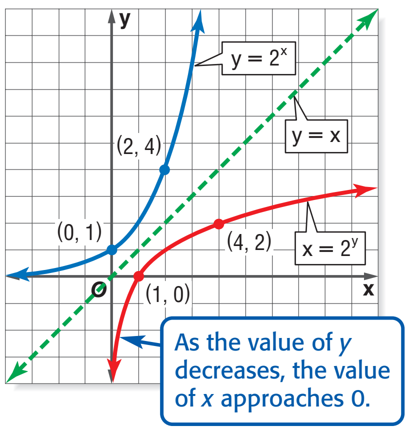
Inverse Function
The inverse function of $f(x)$ is $f^{-1}(x)$. (Lesson 7-6)

Concepts in Motion
Animation
algebra2.com

Logarithmic Functions and Expressions To better understand what is meant by a logarithm, consider the graph of $y = 2^x$ and its inverse. Since exponential functions are one-to-one, the inverse of $y = 2^x$ exists and is also a function. Recall that you can graph the inverse of a function by interchanging the *x*- and *y*-values in the ordered pairs of the function. Consider the exponential function $y = 2^x$.

$y = 2^x$	
x	*y*
−3	$\frac{1}{8}$
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$	
x	*y*
$\frac{1}{8}$	−3
$\frac{1}{4}$	−2
$\frac{1}{2}$	−1
1	0
2	1
4	2
8	3



The inverse of $y = 2^x$ can be defined as $x = 2^y$. Notice that the graphs of these two functions are reflections of each other over the line $y = x$.

In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, y is called the **logarithm** of x. It is usually written as $y = \log_b x$ and is read *y equals log base b of x*.

KEY CONCEPT — *Logarithm with Base b*

Words Let b and x be positive numbers, $b \neq 1$. The *logarithm of x with base b* is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.

Symbols Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.

EXAMPLE 1 Logarithmic to Exponential Form

Study Tip

Zero Exponent
Recall that for any $b \neq 0$, $b^0 = 1$.

Write each equation in exponential form.

a. $\log_8 1 = 0$

$\log_8 1 = 0 \rightarrow 1 = 8^0$

b. $\log_2 \frac{1}{16} = -4$

$\log_2 \frac{1}{16} = -4 \rightarrow \frac{1}{16} = 2^{-4}$

CHECK Your Progress

1A. $\log_4 16 = 2$

1B. $\log_3 \frac{1}{27} = -3$

EXAMPLE 2 Exponential to Logarithmic Form

Write each equation in logarithmic form.

a. $10^3 = 1000$

$10^3 = 1000 \rightarrow \log_{10} 1000 = 3$

b. $9^{\frac{1}{2}} = 3$

$9^{\frac{1}{2}} = 3 \rightarrow \log_9 3 = \frac{1}{2}$

CHECK Your Progress

2A. $4^3 = 64$

2B. $125^{\frac{1}{3}} = 5$

You can use the definition of logarithm to find the value of a logarithmic expression.

EXAMPLE 3 Evaluate Logarithmic Expressions

Evaluate $\log_2 64$.

$\log_2 64 = y$	Let the logarithm equal y.
$64 = 2^y$	Definition of logarithm
$2^6 = 2^y$	$64 = 2^6$
$6 = y$	Property of Equality for Exponential Functions

So, $\log_2 64 = 6$.

CHECK Your Progress

Evaluate each expression.

3A. $\log_3 81$

3B. $\log_4 256$

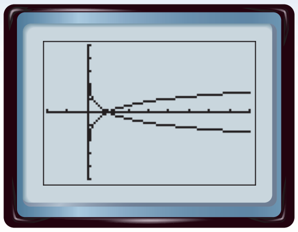
The function $y = \log_b x$, where $b > 0$ and $b \neq 1$, is called a **logarithmic function.** As shown in the graph on the previous page, this function is the inverse of the exponential function $y = b^x$ and has the following characteristics.

1. The function is continuous and one-to-one.
2. The domain is the set of all positive real numbers.
3. The y-axis is an asymptote of the graph.
4. The range is the set of all real numbers.
5. The graph contains the point (1, 0). That is, the x-intercept is 1.

GEOMETRY SOFTWARE LAB

The calculator screen shows the graphs of $y = \log_4 x$ and $y = \log_{\frac{1}{4}} x$.

KEYSTROKES: [Y=] [LOG] [X,T,θ,n] [ )] [÷] [LOG] 4 [ )] [ENTER]
[LOG] [X,T,θ,n] [ )] [÷] [LOG] 1 [÷] 4 [ )] [GRAPH]



THINK AND DISCUSS

1. How do the shapes of the graphs compare?

2. How do the asymptotes and the x-intercepts of the graphs compare?

3. Describe the relationship between the graphs.

4. Graph each pair of functions on the same screen. Then compare and contrast the graphs.

a.	$y = \log_4 x$	$y = \log_4 x + 2$
b.	$y = \log_4 x$	$y = \log_4 (x + 2)$
c.	$y = \log_4 x$	$y = 3 \log_4 x$

5. Describe the relationship between $y = \log_4 x$ and $y = -1(\log_4 x)$.

6. What are a reasonable domain and range for each function?

7. What is a reasonable viewing window in order to see the trends of both functions?

Study Tip

Look Back

To review **composition of functions**, see Lesson 7-5.

Since the exponential function $f(x) = b^x$ and the logarithmic function $g(x) = \log_b x$ are inverses of each other, their composites are the identity function. That is, $f[g(x)] = x$ and $g[f(x)] = x$.

$$f[g(x)] = x \qquad\qquad g[f(x)] = x$$
$$f(\log_b x) = x \qquad\qquad g(b^x) = x$$
$$b^{\log_b x} = x \qquad\qquad \log_b b^x = x$$

Thus, if their bases are the same, exponential and logarithmic functions "undo" each other. You can use this inverse property of exponents and logarithms to simplify expressions and solve equations. For example, $\log_6 6^8 = 8$ and $3^{\log_3 (4x - 1)} = 4x - 1$.

Solve Logarithmic Equations and Inequalities A **logarithmic equation** is an equation that contains one or more logarithms. You can use the definition of a logarithm to help you solve logarithmic equations.

EXAMPLE 4 Solve a Logarithmic Equation

Solve $\log_4 n = \frac{5}{2}$.

$\log_4 n = \frac{5}{2}$ Original equation

$n = 4^{\frac{5}{2}}$ Definition of logarithm

$n = (2^2)^{\frac{5}{2}}$ $4 = 2^2$

$n = 2^5$ or 32 Power of a Power

CHECK Your Progress

Solve each equation.

4A. $\log_9 x = \frac{3}{2}$ **4B.** $\log_{16} x = \frac{5}{2}$

A **logarithmic inequality** is an inequality that involves logarithms. In the case of inequalities, the following property is helpful.

KEY CONCEPT *Logarithmic to Exponential Inequality*

Symbols If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.
If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.

Examples $\log_2 x > 3$, $x > 2^3$ $\log_3 x < 5$, $0 < x < 3^5$

Study Tip

Special Values

If $b > 0$ and $b \neq 1$, then the following statements are true.

- $\log_b b = 1$ because $b^1 = b$.
- $\log_b 1 = 0$ because $b^0 = 1$.

EXAMPLE 5 Solve a Logarithmic Inequality

Solve $\log_5 x < 2$. Check your solution.

$\log_5 x < 2$ Original inequality

$0 < x < 5^2$ Logarithmic to exponential inequality

$0 < x < 25$ Simplify.

The solution set is $\{x \mid 0 < x < 25\}$.

CHECK Try 5 to see if it satisfies the inequality.

$\log_5 x < 2$ Original inequality

$\log_5 5 \overset{?}{<} 2$ Substitute 5 for x.

$1 < 2$ ✓ $\log_5 5 = 1$ because $5^1 = 5$.

CHECK Your Progress

Solve each inequality. Check your solution.

5A. $\log_4 x > 3$ **5B.** $\log_2 x < 4$

Use the following property to solve logarithmic equations that have logarithms with the same base on each side.

KEY CONCEPT — *Property of Equality for Logarithmic Functions*

Symbols If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_7 x = \log_7 3$, then $x = 3$.

EXAMPLE Solve Equations with Logarithms on Each Side

6 **Solve $\log_5 (p^2 - 2) = \log_5 p$. Check your solution.**

$\log_5 (p^2 - 2) = \log_5 p$	Original equation
$p^2 - 2 = p$	Property of Equality for Logarithmic Functions
$p^2 - p - 2 = 0$	Subtract p from each side.
$(p - 2)(p + 1) = 0$	Factor.
$p - 2 = 0$ or $p + 1 = 0$	Zero Product Property
$p = 2$ $\quad p = -1$	Solve each equation.

CHECK Substitute each value into the original equation.

Check $p = 2$.

$\log_5 (2^2 - 2) \stackrel{?}{=} \log_5 2$	Substitute 2 for p.
$\log_5 2 = \log_5 2$ ✓	Simplify.

Check $p = -1$.

$\log_5 [(-1)^2 - 2] \stackrel{?}{=} \log_5 (-1)$	Substitute -1 for p.

Since $\log_5 (-1)$ is undefined, -1 is an *extraneous* solution and must be eliminated. Thus, the solution is 2.

Study Tip

Extraneous Solutions

The domain of a logarithmic function does not include negative values. For this reason, be sure to check for extraneous solutions of logarithmic equations.

CHECK Your Progress

Solve each equation. Check your solution.

6A. $\log_3 (x^2 - 15) = \log_3 2x$ **6B.** $\log_{14} (m^2 - 30) = \log_{14} m$

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Use the following property to solve logarithmic inequalities that have the same base on each side. Exclude values from your solution set that would result in taking the logarithm of a number less than or equal to zero in the original inequality.

KEY CONCEPT — *Property of Inequality for Logarithmic Functions*

Symbols If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

Example If $\log_2 x > \log_2 9$, then $x > 9$.

This property also holds for $\leq$ and $\geq$.

Study Tip

Look Back
To review **compound inequalities**, see Lesson 1-6.

EXAMPLE 7 Solve Inequalities with Logarithms on Each Side

Solve $\log_{10}(3x - 4) < \log_{10}(x + 6)$. Check your solution.

$\log_{10}(3x - 4) < \log_{10}(x + 6)$ — Original inequality

$3x - 4 < x + 6$ — Property of Inequality for Logarithmic Functions

$2x < 10$ — Addition and Subtraction Properties of Inequalities

$x < 5$ — Divide each side by 2.

We must exclude from this solution all values of x such that $3x - 4 \le 0$ or $x + 6 \le 0$.

Thus, $x > \frac{4}{3}$, $x > -6$, and $x < 5$. This compound inequality simplifies to $\frac{4}{3} < x < 5$. The solution set is $\left\{x \,\middle|\, \frac{4}{3} < x < 5\right\}$.

CHECK Your Progress

7. Solve $\log_5(2x + 1) \le \log_5(x + 4)$. Check your solution.

CHECK Your Understanding

Example 1 (p. 510)

Write each equation in logarithmic form.

1. $5^4 = 625$ **2.** $7^{-2} = \frac{1}{49}$ **3.** $3^5 = 243$

Example 2 (p. 510)

Write each equation in exponential form.

4. $\log_3 81 = 4$ **5.** $\log_{36} 6 = \frac{1}{2}$ **6.** $\log_{125} 5 = \frac{1}{3}$

Example 3 (p. 510)

Evaluate each expression.

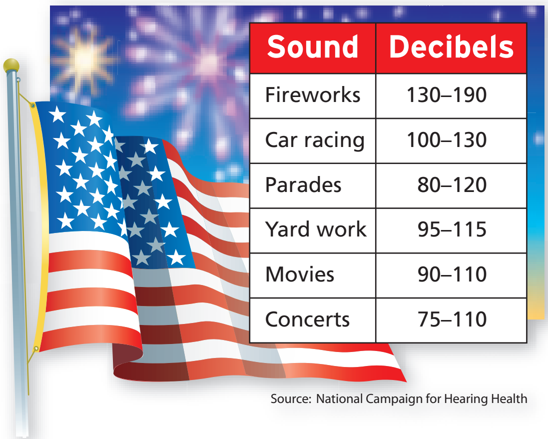
7. $\log_4 256$ **8.** $\log_2 \frac{1}{8}$ **9.** $\log_6 216$

Example 4 (p. 512)

Solve each equation. Check your solutions.

10. $\log_9 x = \frac{3}{2}$ **11.** $\log_{\frac{1}{10}} x = -3$ **12.** $\log_b 9 = 2$

SOUND **For Exercises 13–15, use the following information.**
An equation for loudness L, in decibels, is $L = 10 \log_{10} R$, where R is the relative intensity of the sound.



Sound	Decibels
Fireworks	130–190
Car racing	100–130
Parades	80–120
Yard work	95–115
Movies	90–110
Concerts	75–110

Source: National Campaign for Hearing Health

13. Solve $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels.

14. Solve $75 = 10 \log_{10} R$ to find the relative intensity of a concert with a loudness of 75 decibels.

15. How many times more intense is the fireworks display than the concert? In other words, find the ratio of their intensities.

Example 5 (p. 512) **Solve each inequality. Check your solutions.**

16. $\log_4 x < 2$ **17.** $\log_3 (2x - 1) \leq 2$ **18.** $\log_{16} x \geq \frac{1}{4}$

Example 6 (p. 513) **Solve each equation. Check your solutions.**

19. $\log_5 (3x - 1) = \log_5 (2x^2)$ **20.** $\log_{10} (x^2 - 10x) = \log_{10} (-21)$

Example 7 (p. 514) **Solve each inequality. Check your solutions.**

21. $\log_2 (3x - 5) > \log_2 (x + 7)$ **22.** $\log_5 (5x - 7) \leq \log_5 (2x + 5)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
23–28	1
29–34	2
35–43	3
44–51	4
52–55	5
56, 57	6
58, 59	7

Write each equation in exponential form.

23. $\log_5 125 = 3$ **24.** $\log_{13} 169 = 2$ **25.** $\log_4 \frac{1}{4} = -1$

26. $\log_{100} \frac{1}{10} = -\frac{1}{2}$ **27.** $\log_8 4 = \frac{2}{3}$ **28.** $\log_{\frac{1}{5}} 25 = -2$

Write each equation in logarithmic form.

29. $8^3 = 512$ **30.** $3^3 = 27$ **31.** $5^{-3} = \frac{1}{125}$

32. $\left(\frac{1}{3}\right)^{-2} = 9$ **33.** $100^{\frac{1}{2}} = 10$ **34.** $2401^{\frac{1}{4}} = 7$

Evaluate each expression.

35. $\log_2 16$ **36.** $\log_{12} 144$ **37.** $\log_{16} 4$

38. $\log_9 243$ **39.** $\log_2 \frac{1}{32}$ **40.** $\log_3 \frac{1}{81}$

41. $\log_{10} 0.001$ **42.** $\log_4 16^x$ **43.** $\log_3 27^x$

Solve each equation. Check your solutions.

44. $\log_9 x = 2$ **45.** $\log_{25} n = \frac{3}{2}$ **46.** $\log_{\frac{1}{7}} x = -1$

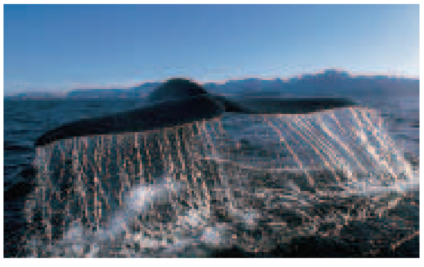
47. $\log_{10} (x^2 + 1) = 1$ **48.** $\log_b 64 = 3$ **49.** $\log_b 121 = 2$

WORLD RECORDS **For Exercises 50 and 51, use the information given for Exercises 13–15 to find the relative intensity of each sound.**

Source: *The Guinness Book of Records*

50. The loudest animal sounds are the low-frequency pulses made by blue whales when they communicate. These pulses have been measured up to 188 decibels.

51. The loudest insect is the African cicada that produces a calling song that measures 106.7 decibels at a distance of 50 centimeters.



EXTRA PRACTICE
See pages 910, 934.

Math Online
Self-Check Quiz at algebra2.com

Solve each equation or inequality. Check your solutions.

52. $\log_2 c > 8$

53. $\log_{64} y \leq \frac{1}{2}$

54. $\log_{\frac{1}{3}} p < 0$

55. $\log_2 (3x - 8) \geq 6$

56. $\log_6 (2x - 3) = \log_6 (x + 2)$

57. $\log_7 (x^2 + 36) = \log_7 100$

58. $\log_2 (4y - 10) \geq \log_2 (y - 1)$

59. $\log_{10} (a^2 - 6) > \log_{10} a$

Show that each statement is true.

60. $\log_5 25 = 2 \log_5 5$

61. $\log_{16} 2 \cdot \log_2 16 = 1$

62. $\log_7 [\log_3 (\log_2 8)] = 0$

63. Sketch the graphs of $y = \log_{\frac{1}{2}} x$ and $y = \left(\frac{1}{2}\right)^x$ on the same axes. Then describe the relationship between the graphs.

64. Sketch the graphs of $y = \log_3 x$, $y = \log_3 (x + 2)$, $y = \log_3 x - 3$. Then describe the relationship between the graphs.

Real-World Link

The Loma Prieta earthquake measured 7.1 on the Richter scale and interrupted the 1989 World Series in San Francisco.

Source: U.S. Geological Survey

EARTHQUAKES **For Exercises 65 and 66, use the following information.**
The magnitude of an earthquake is measured on a logarithmic scale called the Richter scale. The magnitude M is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion.

65. How many times as great is the amplitude caused by an earthquake with a Richter scale rating of 7 as an aftershock with a Richter scale rating of 4?

66. How many times as great was the motion caused by the 1906 San Francisco earthquake that measured 8.3 on the Richter scale as that caused by the 2001 Bhuj, India, earthquake that measured 6.9?

67. **NOISE ORDINANCE** A proposed city ordinance will make it illegal to create sound in a residential area that exceeds 72 decibels during the day and 55 decibels during the night. How many times as intense is the noise level allowed during the day than at night? (*Hint:* See information on page 514.)

Graphing Calculator

FAMILY OF GRAPHS **For Exercises 68 and 69, use the following information.**
Consider the functions $y = \log_2 x + 3$, $y = \log_2 x - 4$, $y = \log_2 (x - 1)$, and $y = \log_2 (x + 2)$.

68. Use a graphing calculator to sketch the graphs on the same screen. Describe this family of graphs in terms of its parent graph $y = \log_2 x$.

69. What are a reasonable domain and range for each function?

H.O.T. Problems

70. **OPEN ENDED** Give an example of an exponential equation and its related logarithmic equation.

71. **Which One Doesn't Belong?** Find the expression that does not belong. Explain.

$\log_4 16$	$\log_2 16$	$\log_2 4$	$\log_3 9$

72. **FIND THE ERROR** Paul and Clemente are solving $\log_3 x = 9$. Who is correct? Explain your reasoning.

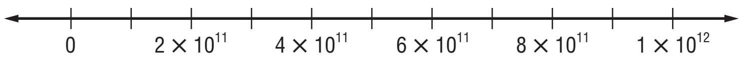
Paul
$\log_3 x = 9$
$3^x = 9$
$3^x = 3^2$
$x = 2$

Clemente
$\log_3 x = 9$
$x = 3^9$
$x = 19{,}683$

David Weintraub/Photo Researchers

73. CHALLENGE Using the definition of a logarithmic function where $y = \log_b x$, explain why the base b cannot equal 1.

74. *Writing in Math* Use the information about sound on page 509 to explain how a logarithmic scale can be used to measure sound. Include the relative intensities of a pin drop, a whisper, normal conversation, kitchen noise, and a jet engine written in scientific notation. Also include a plot of each of these relative intensities on the scale below and an explanation as to why the logarithmic scale might be preferred over the scale below.



STANDARDIZED TEST PRACTICE

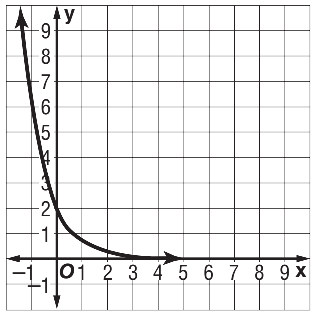
75. ACT/SAT What is the equation of the function?

A $y = 2(3)^x$

B $y = 2\left(\frac{1}{3}\right)^x$

C $y = 3\left(\frac{1}{2}\right)^x$

D $y = 3(2)^x$



76. REVIEW What is the solution to the equation $3^x = 11$?

F $x = 2$

G $x = \log_{10} 2$

H $x = \log_{10} 11 + \log_{10} 3$

J $x = \dfrac{\log_{10} 11}{\log_{10} 3}$

Spiral Review

Simplify each expression. (Lesson 9-1)

77. $x^{\sqrt{6}} \cdot x^{\sqrt{6}}$

78. $\left(b^{\sqrt{6}}\right)^{\sqrt{24}}$

Solve each equation. Check your solutions. (Lesson 8-6)

79. $\dfrac{2x+1}{x} - \dfrac{x+1}{x-4} = \dfrac{-20}{x^2-4x}$

80. $\dfrac{2a-5}{a-9} - \dfrac{a-3}{3a+2} = \dfrac{5}{3a^2-25a-18}$

Solve each equation by using the method of your choice. Find exact solutions. (Lesson 5-6)

81. $9y^2 = 49$

82. $2p^2 = 5p + 6$

83. BANKING Donna Bowers has $8000 she wants to save in the bank. A 12-month certificate of deposit (CD) earns 4% annual interest, while a regular savings account earns 2% annual interest. Ms. Bowers doesn't want to tie up all her money in a CD, but she has decided she wants to earn $240 in interest for the year. How much money should she put in to each type of account? (Lesson 4-4)

GET READY for the Next Lesson

Simplify. Assume that no variable equals zero. (Lesson 6-1)

84. $x^4 \cdot x^6$

85. $(2a^2b)^3$

86. $\dfrac{a^4n^7}{a^3n}$

87. $\left(\dfrac{b^7}{a^4}\right)^0$

EXTEND 9-2

Graphing Calculator Lab
Modeling Data Using Exponential Functions

We are often confronted with data for which we need to find an equation that best fits the information. We can find exponential and logarithmic functions of best fit using a TI-83/84 Plus graphing calculator.

ACTIVITY

The population per square mile in the United States has changed dramatically over a period of years. The table shows the number of people per square mile for several years.

U.S. Population Density

Year	People per square mile	Year	People per square mile
1790	4.5	1900	21.5
1800	6.1	1910	26.0
1810	4.3	1920	29.9
1820	5.5	1930	34.7
1830	7.4	1940	37.2
1840	9.8	1950	42.6
1850	7.9	1960	50.6
1860	10.6	1970	57.5
1870	10.9	1980	64.0
1880	14.2	1990	70.3
1890	17.8	2000	80.0

Source: Northeast-Midwest Institute

a. Use a graphing calculator to enter the data and draw a scatter plot that shows how the number of people per square mile is related to the year.

Step 1 Enter the year into L1 and the people per square mile into L2.

KEYSTROKES: *See pages 92 and 93 to review how to enter lists.*

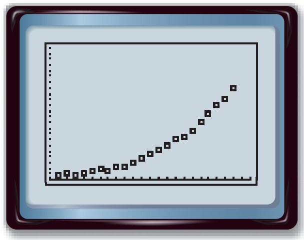
Be sure to clear the Y= list. Use the ▶ key to move the cursor from L1 to L2.

Step 2 Draw the scatter plot.

KEYSTROKES: *See pages 92 and 93 to review how to graph a scatter plot.*

Make sure that Plot 1 is on, the scatter plot is chosen, Xlist is L1, and Ylist is L2. Use the viewing window [1780, 2020] with a scale factor of 10 by [0, 115] with a scale factor of 5.

We see from the graph that the equation that best fits the data is a curve. Based on the shape of the curve, try an exponential model.



[1780, 2020] scl: 10 by [0, 115] scl: 5

Step 3 To determine the exponential equation that best fits the data, use the exponential regression feature of the calculator.

KEYSTROKES: STAT ▶ 0 2nd [L1] , 2nd [L2] ENTER